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# Perturbative Strong Interaction Corrections to the Heavy Quark Semileptonic Decay Rate

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## Abstract

We calculate the part of the order  $\alpha_s^2$  correction to the semileptonic heavy quark decay rate proportional to the number of light quark flavors, and use our result to set the scale for evaluating the strong coupling in the order  $\alpha_s$  term according to the scheme of Brodsky, Lepage and Mackenzie. Expressing the decay rate in terms of the heavy quark pole mass  $m_Q$ , we find the scale for the  $\overline{MS}$  strong coupling to be  $0.07 m_Q$ . If the decay rate is expressed in terms of the  $\overline{MS}$  heavy quark mass  $\overline{m}_Q(m_Q)$  then the scale is  $0.12 m_Q$ . We use these results along with the existing calculations for hadronic  $\tau$  decay to calculate the BLM scale for the nonleptonic decay width and the semileptonic branching ratio. The implications for the value of  $|V_{bc}|$  extracted from the inclusive semileptonic  $B$  meson decay rate are discussed.

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Inclusive semileptonic  $B$  decay has received considerable attention both theoretically and experimentally. In the limit where the  $b$  quark mass is much larger than the QCD scale the  $B$  meson decay rate is equal to the  $b$  quark decay rate [1]. Corrections to this first arise at order  $(\Lambda_{QCD}/m_b)^2$  and these nonperturbative corrections may be written in terms of the matrix elements [2–4]  $\langle B|\bar{b}(iD)^2b|B\rangle$  and  $\langle B|\bar{b}ig\sigma_{\mu\nu}G^{\mu\nu}b|B\rangle$ . The measured semileptonic  $B$  decay rate provides a method for determining the magnitude of the element of the Cabibbo–Kobayashi–Maskawa matrix  $V_{cb}$ . To one loop, the  $b$  quark decay rate is

$$\Gamma(b \rightarrow X_c e \bar{\nu}_e) = |V_{cb}|^2 \frac{G_F^2 m_b^5}{192\pi^3} f(m_c/m_b) \times \left[ 1 - \frac{2\alpha_s}{3\pi} \left( \pi^2 - \frac{25}{4} + \delta_1(m_c/m_b) \right) + \dots \right]. \quad (1)$$

In Eq. (1)  $m_b$  and  $m_c$  are the pole masses of the  $b$  and  $c$  quarks,  $f(x)$  is defined by

$$f(x) = (1 - x^4)(1 - 8x^2 + x^4) - 24x^4 \ln x \quad (2)$$

and  $\delta_1$  takes into account the effects of the charm quark mass on the order  $\alpha_s$  contribution to the  $b$  quark decay rate [6]. For  $m_c/m_b = 0.3$ ,  $\delta_1 = -1.11$ .

In Eq. (1) the scale of the strong coupling  $\alpha_s$  is usually taken to be  $\sim m_b$ . The size of the order  $\alpha_s$  correction depends critically on this choice. If all of the higher order terms in the  $\alpha_s$  expansion were known then the decay rate would be independent of the choice of scale. However, some choices of scale give perturbation series that are badly behaved with higher orders in the coupling being very important. Brodsky, Lepage and Mackenzie (BLM) [5] have advocated choosing the scale so that vacuum polarization effects are absorbed into the running coupling. This physically appealing choice of scale usually results in a reasonable perturbation series. In this letter we use our calculation of the part of the  $\alpha_s^2$  correction proportional to  $n_f$  to determine the BLM scale appropriate for semileptonic heavy quark decay.

Smith and Voloshin [7] have recently shown that the  $n_f$  dependent part of the order  $\alpha_s^2$  contribution to the semileptonic decay rate for a heavy quark may be written in terms of the one loop corrections evaluated with a finite gluon mass:

$$\delta\Gamma^{(2)} = \frac{n_f \alpha_s^{(V)}(m_Q)}{6\pi} \int_0^\infty \left( \Gamma^{(1)}(\mu) - \frac{m_Q^2}{(\mu^2 + m_Q^2)} \Gamma^{(1)}(0) \right) \frac{d\mu^2}{\mu^2} \quad (3)$$

where  $\Gamma^{(1)}(\mu)$  is the order  $\alpha_s$  contribution to the decay rate computed with a gluon of mass  $\mu$ , and  $\alpha_s^{(V)}(m_Q)$  is the strong coupling evaluated in the  $V$ -scheme of Brodsky, Lepage and Mackenzie. This is related to the usual  $\overline{MS}$  coupling  $\bar{\alpha}_s(m_b)$  by [5]

$$\alpha_s^{(V)}(\mu) = \bar{\alpha}_s(\mu) + \frac{5}{3} \frac{\bar{\alpha}_s^2(\mu)}{4\pi} \left( 11 - \frac{2}{3} n_f \right) + \dots \quad (4)$$

An expression analogous to Eq. (3) also holds for the differential rate  $d\Gamma/dt$ , where  $t = (p_e + p_{\bar{\nu}})^2$ . We have calculated  $\delta\Gamma^{(2)}$  for semileptonic  $Q \rightarrow X_q e \bar{\nu}_e$  decay with a massless quark  $q$  in the final state and  $m_Q \ll m_W$ . The contribution of the graphs containing a virtual gluon loop to the differential rate  $d\Gamma/dt$  with a massive gluon was calculated analytically while the integral over the  $c$  quark energy in the bremsstrahlung graphs was performed numerically. The infrared divergences were shown explicitly to cancel in the sum, and the final integral over the gluon mass was performed numerically. Finally, the  $t$  integral was also performed numerically to obtain the correction to the total rate. In the  $t \rightarrow 0$  limit we reproduce the results given in [7] for top quark decay in the limit  $m_t \gg m_W$ .

We find for the total decay width

$$\Gamma(Q \rightarrow X_q e \bar{\nu}_e) = |V_{Qq}|^2 \frac{G_F^2 m_Q^5}{192\pi^3} \left\{ 1 - \frac{2\alpha_s^{(V)}(m_Q)}{3\pi} \left( \pi^2 - \frac{25}{4} \right) + \frac{2}{3} n_f \left( \frac{\alpha_s^{(V)}(m_Q)}{\pi} \right)^2 [2.22] + \dots \right\}. \quad (5)$$

In terms of the  $\overline{MS}$  coupling Eq. (5) becomes

$$\Gamma(Q \rightarrow X_q e \bar{\nu}_e) = |V_{Qq}|^2 \frac{G_F^2 m_Q^5}{192\pi^3} \left\{ 1 - \frac{2\bar{\alpha}_s(m_Q)}{3\pi} \left( \pi^2 - \frac{25}{4} \right) + \frac{2}{3} n_f \left( \frac{\bar{\alpha}_s(m_Q)}{\pi} \right)^2 [3.22] + \dots \right\}. \quad (6)$$

In Eq. (6) the term proportional to  $n_f$  can be absorbed into the order  $\bar{\alpha}_s$  term if the scale is changed from  $m_Q$  to  $\mu_{\text{BLM}}$ , where

$$\mu_{\text{BLM}} = m_Q \exp \left\{ \frac{-3}{(\pi^2 - 25/4)} [3.22] \right\} \simeq 0.07 m_Q. \quad (7)$$

The BLM scale for inclusive heavy quark decay is therefore significantly smaller than the naïve estimate of  $m_Q$ . In Fig. 1 we plot the BLM scale for the differential rate  $d\Gamma/dt$  as a function of the squared invariant mass  $t$  of the lepton pair. At  $t = 0$  we find the scale  $\mu_{\text{BLM}} = 0.12 m_Q$ , which coincides with the BLM scale found in [7] for top quark decay in the limit  $m_W \ll m_t$ . As would be expected on physical grounds,  $\mu_{\text{BLM}}$  decreases as the invariant mass of the lepton pair increases.

The expression for the width found in Eq. (6) is given in terms of the pole mass  $m_Q$  of the heavy quark. The BLM scale  $\mu_{\text{BLM}}$  is different from that found in Eq. (7) if the rate is expressed in terms of the running  $\overline{MS}$  heavy quark mass evaluated at  $m_Q$ . Using [8]

$$m_Q = \bar{m}_Q(m_Q) \left\{ 1 + \frac{4}{3} \frac{\bar{\alpha}_s(m_Q)}{\pi} + (16.11 - 1.04n_f) \left( \frac{\bar{\alpha}_s(m_Q)}{\pi} \right)^2 + \dots \right\} \quad (8)$$

the semileptonic decay rate becomes

$$\Gamma(Q \rightarrow X_q e \bar{\nu}_e) = |V_{Qq}|^2 \frac{G_F^2 [\bar{m}_Q(m_Q)]^5}{192\pi^3} \left\{ 1 - \frac{2\bar{\alpha}_s(m_Q)}{3\pi} \left( \pi^2 - \frac{65}{4} \right) + \frac{2}{3} n_f \left( \frac{\bar{\alpha}_s(m_Q)}{\pi} \right)^2 [-4.58] + \dots \right\}. \quad (9)$$

Now the scale  $\mu_{\text{BLM}}$  for which vacuum polarization effects are absorbed into the strong coupling is

$$\mu_{\text{BLM}} = m_Q \exp \left\{ \frac{-3}{(\pi^2 - 65/4)} [-4.58] \right\} \simeq 0.12 m_Q. \quad (10)$$

It has been argued [7] that a low BLM scale, indicating large two-loop corrections when  $\alpha_s(m_Q)$  is used as an expansion parameter, would be expected when relating a “long-distance” quantity such as the heavy quark pole mass to the “short-distance” decay rate. However, our results show that even if the “short-distance”  $\overline{MS}$  heavy quark mass is used, the BLM scale  $\mu_{\text{BLM}}$  for the order  $\bar{\alpha}_s$  correction to semileptonic heavy quark decay is still significantly less than  $m_Q$ . For  $b$  decay (neglecting the charm quark mass) the scale is about 500 MeV and for  $c$  decay rate it is only about 150 MeV. These low scales suggest that QCD perturbation theory cannot be used for inclusive semileptonic  $D$  or  $\Lambda_c$  decay and that an accurate extraction of  $|V_{cb}|$  from the inclusive semileptonic  $B$  decay rate [9] is not possible

without including all terms of order  $\bar{\alpha}_s^2(m_Q)$  (and perhaps even higher orders in  $\bar{\alpha}_s$ ) in the theoretical expression for the semileptonic decay rate.

We also note that the BLM scale for inclusive semileptonic heavy quark decay is somewhat smaller (relative to the heavy quark mass) than the analogous scale for hadronic  $\tau$  decay. From the two-loop expression for the inclusive  $\tau$  width [10],

$$\frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau \bar{\nu}_e e^-)} = 3 \left( 1 + \frac{\bar{\alpha}_s(m_\tau)}{\pi} + (6.340 - 0.379n_f) \left( \frac{\bar{\alpha}_s(m_\tau)}{\pi} \right)^2 + \dots \right) \quad (11)$$

the BLM scale for the one-loop expression is found to be  $\exp(-3 \times 0.379)m_\tau = 0.32 m_\tau$ . Therefore, although inclusive  $\tau$  and  $c$  decays involve comparable energy scales, perturbative QCD is likely to be at best applicable only to the former.

It is straightforward to extend our results to the case of nonleptonic heavy quark decay to massless products if we neglect the running of the Hamiltonian between  $m_W$  and  $\mu$ . At order  $\alpha_s$ , the corrections to the nonleptonic width are given by two classes of diagrams: those with gluons dressing the  $\bar{c}b$  vertex and those with gluons dressing the  $\bar{d}u$  vertex. The first class is identical to that encountered in semileptonic decay, while the second gives the corrections to  $\tau$  decay. Combining Eqs. (11) and (6) and including the appropriate colour factors, we find the expression for the total nonleptonic width to massless quarks

$$\Gamma_{nl} \equiv \Gamma(Q \rightarrow X_{q_1} X_{q_2} X_{\bar{q}_3}) = |V_{Qq_1}|^2 |V_{q_2 q_3}|^2 \frac{G_F^2 m_Q^5}{64\pi^3} \left\{ 1 - \frac{2\bar{\alpha}_s(m_Q)}{3\pi} \left( \pi^2 - \frac{31}{4} \right) + \frac{2}{3} n_f \left( \frac{\bar{\alpha}_s(m_Q)}{\pi} \right)^2 [2.65] + \dots \right\}. \quad (12)$$

Because the one-loop correction to  $\Gamma_{nl}$  is much smaller than for the semileptonic width  $\Gamma_{sl}$ , while the order  $n_f \alpha_s^2$  terms are comparable, this results in a very low BLM scale for the nonleptonic width:

$$\mu_{\text{BLM}} = m_Q \exp \left\{ \frac{-3}{(\pi^2 - 31/4)} [2.65] \right\} \simeq 0.02 m_Q. \quad (13)$$

Such a low scale should not be taken literally. It simply indicates that the two-loop corrections to  $\Gamma_{nl}$  are significant, requiring an extremely low scale for  $\alpha_s$  in the one-loop term to absorb the order  $n_f \alpha_s^2$  terms.

We may also set the BLM scale for the semileptonic branching fraction for decays to massless fermions [11]. The contribution from the class of graphs dressing the  $\bar{b}c$  vertex cancels in the ratio  $\Gamma_{sl}/(\Gamma_{nl} + \Gamma_{sl})$ , and the corrections are given solely by the graphs which contribute to  $\tau$  decay\*. We thus find, independent of the our results for  $\Gamma_{sl}$ ,

$$\frac{\Gamma_{sl}}{\Gamma_{nl} + \Gamma_{sl}} = \frac{1}{4} \left( 1 - \frac{3}{4} \left[ \frac{\alpha_s(m_Q)}{\pi} - 0.379n_f \left( \frac{\alpha_s(m_Q)}{\pi} \right)^2 + \dots \right] \right) \quad (14)$$

(taking  $|V_{q_2q_3}| = 1$ ), which gives the same BLM scale relative to  $m_Q$  as in  $\tau$  decay:  $\mu_{BLM} = 0.32 m_Q$ .

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\*This was also recently noted in [12].

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## Figure Captions

1. The BLM scale for the partial width  $d\Gamma/dt$  (in terms of the pole mass  $m_Q$ ) as a function of the lepton pair invariant mass squared  $t$ .

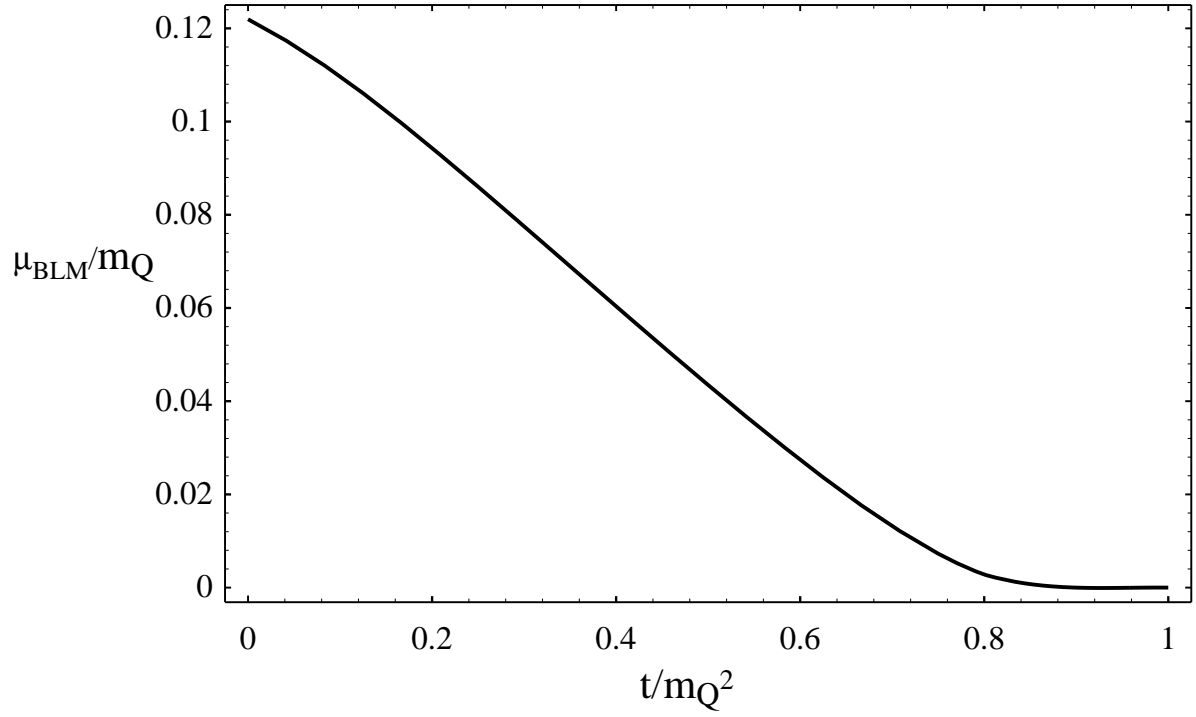


Figure 1



This figure "fig1-1.png" is available in "png" format from:

<http://arXiv.org/ps/hep-ph/9409287v2>